

CONTINUED FRACTIONS ON THE HEISENBERG GROUP

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ABSTRACT. We provide a generalization of continued fractions to the Heisenberg group. We prove the ergodicity of the corresponding Gauss map and the convergence of the infinite continued fraction including an explicit estimate on the rate of convergence. Along the way, we show several surprising analogs of classical formulas about continued fractions.

1. INTRODUCTION

A regular continued fraction (RCF) expansion represents an irrational number $x \in \mathbb{R}$ as

$$(1.1) \quad x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}, \quad a_0 \in \mathbb{Z}, \quad a_i \in \mathbb{N}, i \geq 1.$$

The integers $CF(x) := \{a_0, \dots\}$ are the *continued fraction digits* of x (also called the *partial quotients*). Regular continued fractions and their many variations have played an important part in Diophantine approximation, hyperbolic geometry, and the study of quadratic irrationals.

Many higher-dimensional generalizations of continued fractions have been developed to extend this powerful theory, but these efforts have met with varying success. In this paper, we develop a notion of continued fractions in the non-commutative setting of the Heisenberg group (in a sense, a *complex* two-dimensional continued fraction). Surprisingly, we recover not only standard results of convergence and ergodicity (see Theorems 1.3 and 1.4), but also several simple, direct analogs of classical formulas for regular continued fractions—formulas which lack simple analogs for any other known multi-dimensional continued fraction. This suggests that continued fractions are a reasonable and natural object of study on the Heisenberg group.

This paper provides the basic properties of Heisenberg continued fractions, and opens up the way for many new questions. In future papers, we intend to link our study to that of Diophantine approximation on the Heisenberg group (see [7]) and the behavior of geodesics in complex hyperbolic space (see Remark 2.8 and [16]). Additional interesting questions include extending these results to similar spaces and their lattices (specifically, we expect our results to hold for all boundaries of

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hyperbolic spaces), a characterization of periodic continued fraction expansions, a careful analysis of the Gauss–Kuzmin problem, and a description of the dual space.

The setting for this paper will be the Heisenberg group \mathbb{H} , arguably the most natural non-commutative generalization of \mathbb{R} . Specifically, \mathbb{H} is \mathbb{R}^3 with the modified group law (which we denote by $*$)

$$(1.2) \quad (x, y, t) * (x', y', t') = (x + x', y + y', t + t' + 2(xy' - yx')).$$

Note that in the first two coordinates one sees the usual addition of vectors, while the third coordinate incorporates an antisymmetric term. Note also that the group inverse $(x, y, t)^{-1}$ of an element $(x, y, t) \in \mathbb{H}$ is given by $(-x, -y, -t)$.

Let $\mathbb{H}(\mathbb{Z})$ denote the set of points in \mathbb{H} with all integer coordinates. These form a subgroup of \mathbb{H} , and we will think of them as the integers within \mathbb{H} . Likewise, we think of points with all rational coordinates, $\mathbb{H}(\mathbb{Q})$, as rational points.

Given a generic point $h \in \mathbb{H}$ there is a unique nearest Heisenberg integer $[h] \in \mathbb{H}(\mathbb{Z})$, with respect to the Heisenberg group's standard *gauge metric*:

$$(1.3) \quad \|(x, y, t)\| = \sqrt[4]{(x^2 + y^2)^2 + t^2} \quad d(h, k) = \|h^{-1} * k\|.$$

Note that *left* translations by elements of \mathbb{H} are isometries. That is, $d(g * h, g * k) = d(h, k)$ for all $g, h, k \in \mathbb{H}$. In addition, one has an inversion operation (see §2.1) $\iota : \mathbb{H} \setminus \{0\} \rightarrow \mathbb{H} \setminus \{0\}$ satisfying

$$\|\iota(h)\| = \|h\|^{-1}.$$

Given a point $h \in H$, we may remove the integer part of h via $[h]^{-1} * h$. We may then invert to uncover additional digits.

Definition 1.1. The *continued fraction digits* $CF(h) = \{\gamma_i\}$ and *forward iterates* $\{h_i\}$ of a point $h \in \mathbb{H}$ are defined inductively by:

$$\begin{aligned} \gamma_0 &= [h] & h_0 &= \gamma_0^{-1} * h, \\ \gamma_{i+1} &= [\iota(h_i)] & h_{i+1} &= \gamma_{i+1}^{-1} * \iota(h_i). \end{aligned}$$

Note that $\iota(0)$ is undefined. Thus, the process may terminate after finitely many steps. We will characterize points for which this happens in Theorem 3.10 and, for the majority of the paper, focus our attention on points with infinitely many digits. We will also generally assume that $\gamma_0 = 0$ unless otherwise specified.

Definition 1.2. Let $\{\gamma_i\}$ be a sequence of elements of $\mathbb{H}(\mathbb{Z})$. For a finite sequence, define the associated continued fraction,

$$\mathbb{K}\{\gamma_i\} = \mathbb{K}\{\gamma_i\}_{i=0}^n := \gamma_0 \iota \gamma_1 \iota \cdots \iota \gamma_n,$$

suppressing product notation and parentheses. It is clear that if $CF(h)$ is finite, then $\mathbb{K}CF(h) = h$.

For an infinite sequence, we write

$$\mathbb{K}\{\gamma_i\} = \mathbb{K}\{\gamma_i\}_{i=0}^\infty := \lim_{n \rightarrow \infty} \mathbb{K}\{\gamma_i\}_{i=0}^n,$$

provided the limit exists.

Our main result is to show that \mathbb{K} and CF define a valid notion of a continued fraction expansion for a point in \mathbb{H} . Namely, we prove

Theorem 1.3. *Let $h \in \mathbb{H}$ and $\{\gamma_i\}$ a sequence of elements of $\mathbb{H}(\mathbb{Z})$. Then:*

- (1) *If $\|\gamma_i\| > 3$ for each i , then $\mathbb{K}\{\gamma_i\}$ exists and is unique regardless of whether $\{\gamma_i\}$ is finite or infinite (Theorem 3.7).*
- (2) *A point $h \in \mathbb{H}$ satisfies $h = \mathbb{K}\{\gamma_i\}_{i=0}^n$ if and only if $h \in \mathbb{H}(\mathbb{Q})$ (Theorem 3.10).*
- (3) *Every point in \mathbb{H} has a continued fraction expansion. That is, for all $h \in \mathbb{H}$, the limit $\mathbb{K}CF(h)$ is unique and equal to h (Theorem 3.21).*

Throughout §3, we obtain variants of classical continued fraction results. We show a relationship between the denominator of a rational point and the length of its continued fraction expansion in Theorem 3.11. We find a recursive formula for the approximants $\mathbb{K}\{\gamma_i\}_{i=1}^n$ in Theorem 3.18, and show that the distance between $h \in \mathbb{H}$ and its approximants $\mathbb{K}\{\gamma_i\}_{i=1}^n$ satisfies a variant of a classical relation in Theorem 3.23. We prove that the convergence of $\mathbb{K}CF(h)$ is uniform in Theorem 3.26.

In §4 we consider a generalization of the classical Gauss map $x \mapsto 1/x - \lfloor 1/x \rfloor$. Namely, let $K_D \subset \mathbb{H}$ be the Dirichlet region for $\mathbb{H}(\mathbb{Z})$, defined as the set of points h such that $[h] = 0$. It is easy to see that K_D is a fundamental region for $\mathbb{H}(\mathbb{Z})$, that is, the translates of K_D by elements of $\mathbb{H}(\mathbb{Z})$ tile \mathbb{H} without overlap. It is also clear that for all $h \in \mathbb{H}$, one has $[h]^{-1}h \in K_D$.

We define a function $T : K_D \rightarrow K_D$ on the Dirichlet region by $T(h) = [\iota h]^{-1}\iota h$. We prove (see Theorem 4.10):

Theorem 1.4. *There exists a measure μ on K_D such that:*

- (1) *μ is T -invariant.*
- (2) *μ is absolutely continuous with respect to Lebesgue measure on \mathbb{H} .*
- (3) *T is ergodic with respect to μ .*

More generally, Theorem 4.10 implies the existence of a Gauss–Kuzmin measure on any fundamental domain K for $\mathbb{H}(\mathbb{Z})$ whose closure is strictly contained in the unit sphere (with appropriately modified Gauss map T). In particular, Theorem 1.4 will provide a new measure on the unit cube $K_C = [-1/2, 1/2]^3$, which is also a fundamental domain for $\mathbb{H}(\mathbb{Z})$. See Figure 4 for an illustration of the measures on K_C and K_D .

We will now recall some background on classical continued fractions (§1.1) and the Heisenberg group (§2), and then study Heisenberg continued fractions in §3 and the Gauss map in §4.

1.1. Classical Theory of Continued Fractions. There are many variants on classical continued fractions and many ways to approach them (for good general references, see [3, 6, 8, 11]). We shall examine Nakada’s α -continued fractions, since the study of them bears the most immediate resemblance to the Heisenberg continued fractions we examine in this paper. The α -continued fractions have two well-known continued fraction variants as special cases: Regular Continued Fractions (when $\alpha = 1$) and Nearest Integer Continued Fractions (when $\alpha = 1/2$).

Let $\alpha \in (0, 1]$. Define the α -Gauss map $T_\alpha : [\alpha - 1, \alpha) \rightarrow [\alpha - 1, \alpha)$ by

$$T_\alpha x := \begin{cases} x^{-1} - [x^{-1}]_\alpha, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where $[x]_\alpha$ is the unique integer such that $x - [x]_\alpha \in [\alpha - 1, \alpha)$. Most continued fractions variants begin with these three simple pieces: a fundamental domain ($[\alpha - 1, \alpha)$ here), an inversion that takes a point out of the fundamental domain (x^{-1}), and a linear translation that shifts us back into the fundamental domain ($-[x^{-1}]_\alpha$).

The digits of the α -continued fraction expansion for a number $x \in [\alpha - 1, \alpha)$ consist of two parts, (a_n, ϵ_n) , where

$$a_n = a_n(x) = [T_\alpha^{n-1} x]_\alpha \quad \text{and} \quad \epsilon_n = \epsilon_n(x) = \text{sgn}(T_\alpha^{n-1} x).$$

The sequence of digits (a_n, ϵ_n) terminates when $T_\alpha^n x = 0$. These digits serve to record the data that is lost by iterating the non-injective map T_α . In particular, we have

$$x = \frac{\epsilon_1}{a_1 + T_\alpha x} = \frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + T_\alpha^2 x}} = \dots.$$

Note that $(a_n(x), \epsilon_n(x)) = (a_{n-1}(T_\alpha x), \epsilon_{n-1}(T_\alpha x))$, so that T_α acts as a forward shift of the continued fraction digits of x .

One of the fundamental objects of study in the field of continued fractions is the sequence of *convergents* or *rational approximants* for a number x , given by

$$\frac{p_n}{q_n} := \frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + \dots + \frac{\epsilon_n}{a_n}}}.$$

It is often easier to understand abstract properties of the sequence of convergents for a number x , than it is to understand abstract properties of the whole continued fraction expansion for x .

A particularly useful property of convergents is the following matrix relation:

$$(1.4) \quad \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & \epsilon_2 \\ 1 & a_2 \end{pmatrix} \dots \begin{pmatrix} 0 & \epsilon_n \\ 1 & a_n \end{pmatrix}.$$

From this relation, one can derive the recurrence relation $q_n = a_n q_{n-1} + \epsilon_n q_{n-2}$. While it would be nice to know that the q_n form an increasing sequence of positive integers, this is not always the case (such as with continued fractions with odd partial quotients [1]).

We can treat matrices as Möbius transforms, via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

If we do this, then the simple relation

$$x = \frac{\epsilon_1}{a_1 + T_\alpha x} = \begin{pmatrix} 0 & \epsilon_1 \\ 1 & a_1 \end{pmatrix} T_\alpha x,$$

together with (1.4), implies the more interesting relation

$$(1.5) \quad x = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} T_\alpha^n x = \frac{p_{n-1}T_\alpha^n x + p_n}{q_{n-1}T_\alpha^n x + q_n}.$$

By solving for $T_\alpha^n x$ (or by applying the inverse of the matrix to both sides), one can obtain

$$(1.6) \quad T_\alpha^n x = (-1) \cdot \frac{q_n x - p_n}{q_{n-1} x - p_{n-1}}.$$

Careful—but elementary—manipulation of the formulas (1.5) and (1.6) yields

$$(1.7) \quad q_n x - p_n = (-1)^n \prod_{i=0}^n T_\alpha^n x = (-1)^n \cdot \frac{\epsilon_1 \epsilon_2 \cdots \epsilon_n}{q_n T_\alpha^{n+1} x + q_{n+1}}.$$

From (1.7) it is short exercise to see that $q_n x - p_n$ converges to 0, and hence that p_n/q_n converges to x . Thus it makes sense to write x as an infinite continued fraction expansion

$$x = \frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + \dots}}.$$

There are varying notions of convergence for continued fractions variants besides the fact that $|x - (p_n/q_n)|$ tends to 0, which is typically known as *weak convergence*. In multi-dimensional continued fractions, where one might have convergents

$$\left(\frac{p_{1,n}}{q_n}, \frac{p_{2,n}}{q_n}, \dots, \frac{p_{k,n}}{q_n} \right) \text{ to a point } (x_1, x_2, \dots, x_k),$$

the property that $|q_n x_i - p_{i,n}|$ tends to 0 for all i is known as *strong convergence*. (The Jacobi-Perron continued fraction, which is in many ways considered to be the prototypical multi-dimensional continued fraction, does not satisfy strong convergence.) The fact that all columns of the matrices (1.4) converge (projectively) to the same point is known as *uniform convergence*. Uniform convergence is non-trivial for higher-dimensional continued fraction variants.

In general, it is hard to know whether an arbitrary sequence of continued fraction digits $(a_n, \epsilon_n) \in \mathbb{R}^2$ produces a convergent infinite continued fraction. (Even the seemingly innocuous two-digit sequence $\{(1, 1), (1, -1)\}$ causes convergence problems.) One major result on this question is Pringsheim's theorem, which states that if $|a_n| \geq |\epsilon_n| + 1$ for all $n \in \mathbb{N}$, then the infinite continued fraction converges. For more on this topic, see [17].

For many continued fractions, the digit shift map T is ergodic with respect to some invariant measure. For Regular Continued Fractions ($\alpha = 1$), the invariant measure that is absolutely continuous with respect to Lebesgue is the classic Gauss measure

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx.$$

The ergodicity of the map T means that there is a notion of average behavior for continued fractions. For example, for almost all x , the regular continued fraction

expansion of x satisfies

$$\lim_{n \rightarrow \infty} \frac{\log q_n(x)}{n} = \frac{\pi^2}{12 \log 2}$$

and $a_n = 1$ approximately 42 percent of the time.

Applications of continued fractions come from various areas. We mention only a few in greater detail here. One of the most classical results on continued fractions is Lagrange's Theorem, which states that x has an eventually periodic continued fraction expansion if and only if x is a quadratic irrational number: thus one often studies properties of quadratic irrationals by understanding their RCF expansion. The term $q_n x - p_n$ that appeared in (1.7) is closely related to the study of best approximants—namely, rational numbers n/m that satisfy the following relation

$$|mx - n| \leq |bx - a| \quad a, b \in \mathbb{Z}, \quad 1 \leq b < m$$

must be an RCF convergent p_n/q_n for x .

2. THE HEISENBERG GROUP

We will think of the Heisenberg group in three different ways. For geometric purposes, including illustration and discussion of measures, we will identify \mathbb{H} with \mathbb{R}^3 (with the appropriate group structure and geometry). For the majority of the paper, however, we will be concerned with the representation of \mathbb{H} as a group of unitary matrices or as a subset of \mathbb{C}^2 . This is in direct analogy with thinking of the real numbers as elements of $SL(2, \mathbb{R})$ or as the real axis within \mathbb{C}^1 . We now discuss these models, and then record some information on discrete subgroups of \mathbb{H} and their fundamental domains.

We emphasize that the topological and measure-theoretic notions we consider do not (qualitatively) depend on the model we choose, nor on the metric. In particular, convergence in \mathbb{H} can be shown using the intrinsic gauge metric, or using metrics intrinsic to the model, such as the Euclidean metrics on \mathbb{R}^3 or \mathbb{C}^2 .

2.1. Geometric Model. In the introduction, we defined \mathbb{H} as the space \mathbb{R}^3 with group law

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + 2(xy' - yx')).$$

Combining the first two coordinates into a complex number, \mathbb{H} becomes $\mathbb{C} \times \mathbb{R}$ with group law

$$(z, t) * (z', t') = (z + z', t + t' + 2\operatorname{Im}(\bar{z}z')).$$

We will think of these as the same model, and use it primarily when geometry or visualization are concerned. There are several standard (topologically equivalent) metrics on \mathbb{H} ; we will work with the gauge metric. The gauge $\|\cdot\|$ and distance d are defined by:

$$\|(z, t)\| = \sqrt[4]{|z|^4 + t^2} \quad d(h, k) = \|h^{-1} * k\|.$$

There are four basic transformations we are interested in:

- (1) Left translations $h \mapsto k * h$, for $k \in \mathbb{H}$,

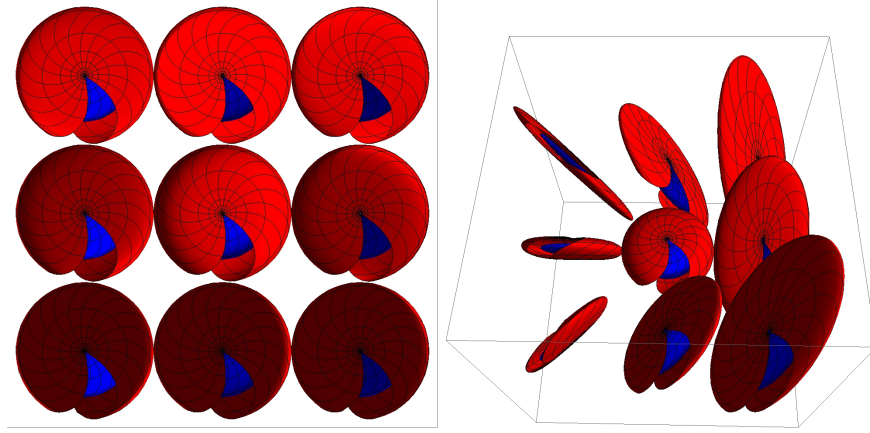


FIGURE 1. Two views of nested spheres in \mathbb{H} , centered at $(i, j, 0)$ with $i, j \in \{-1, 0, 1\}$, related to each other by left translation by elements of $\mathbb{H}(\mathbb{Z})$. In the top view (left), the spheres look identical. A side view (right) shows an additional a shear in the t coordinate.

- (2) Rotations $z \mapsto e^{i\theta}z$, for $\theta \in \mathbb{R}$,
- (3) Metric dilations $\delta_r : (z, t) \mapsto (rz, r^2t)$, for $r \in \mathbb{R}$,
- (4) The Koranyi inversion $\iota : \mathbb{H} \setminus \{0\} \rightarrow \mathbb{H} \setminus \{0\}$ given by

$$\iota(z, t) = \left(\frac{-z}{|z|^2 + it}, \frac{-t}{|z|^4 + t^2} \right).$$

Translations and rotations do not distort distances or volume (that is, the Lebesgue measure λ on \mathbb{R}^3). The map δ_r is a group homomorphism dilating distances by a factor of r and volume by a factor of r^4 . The Koranyi inversion is a conformal map with the following important property.

Lemma 2.1 (See p.19 of [2]). *Let $h, k \in \mathbb{H} \setminus \{0\}$. One has*

$$d(\iota h, \iota k) = \frac{d(h, k)}{\|h\| \|k\|}.$$

In particular, one has $\|\iota h\| = \|h\|^{-1}$, so that the inside and outside of the unit ball are interchanged. Note that individual points on the unit sphere are not fixed.

Remark 2.2. We will show in Lemma 2.12 that ι has a particularly simple form in the unitary model. It is also conformal with respect to the gauge metric [9].

We record the following relationship between volumes and radii of balls in \mathbb{H} . In particular, the lemma implies that the Heisenberg group has Hausdorff dimension 4, and that λ is equivalent to the Hausdorff 4-measure on \mathbb{H} .

Lemma 2.3. *The volume of a ball $B(h, r)$ of radius r around a point h is given by*

$$\lambda(B(h, r)) = r^4 \lambda(B(0, 1))$$

Proof. Applying a left translation, we may assume h is the origin. Further, we may rescale $B(0, r)$ by the Heisenberg dilation $\delta_{1/r}(z, t) = (z/r, t/r^2)$ to obtain $B(0, 1)$. The dilation distorts λ by a factor of r^{-4} . \square

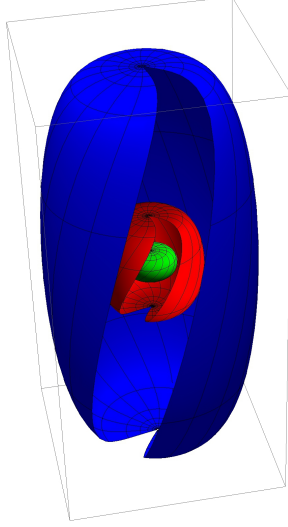


FIGURE 2. Spheres in \mathbb{H} centered at the origin, with radius 2, 1, 1/2, with sectors removed to display nested spheres. The spheres are parametrized by applying ι to a plane; the radial lines of the plane provide the characteristic foliation on the spheres.

2.2. Real Nilpotent Model. It is common to describe \mathbb{H} as the group of nilpotent upper-triangular 3-by-3 real matrices. Our definition is related to this *real nilpotent model* via the Lie group isomorphism

$$(2.1) \quad (x, y, t) \mapsto \begin{pmatrix} 1 & x & \frac{t}{4} + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

We will not use the real nilpotent model, although our results can be rephrased for it. Note that under (2.1), $\mathbb{H}(\mathbb{Z})$ is *not* identified with matrices with integer entries.

2.3. Unitary Representation. For calculation purposes, we will use the (*Siegel*) *unitary representation* of \mathbb{H} . Namely, we will embed \mathbb{H} in $GL(3, \mathbb{C})$ via the homomorphism:

$$(2.2) \quad \mathbb{U} : (z, t) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ z(1 + \mathbf{i}) & 1 & 0 \\ |z|^2 + t\mathbf{i} & \bar{z}(1 - \mathbf{i}) & 1 \end{pmatrix}.$$

Remark 2.4. In literature, one sees a factor of $\sqrt{2}$ rather than $1 + \mathbf{i}$ in the embedding. The latter is more convenient for our purposes.

Let \mathbb{J} be the Hermitian inner product given by

$$\mathbb{J}((z_0, z_1, z_2), (w_0, w_1, w_2)) = \begin{pmatrix} \overline{z_0} & \overline{z_1} & \overline{z_2} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}$$

In particular, we record

$$(2.3) \quad |(z_0, z_1, z_2)|_{\mathbb{J}}^2 = \mathbb{J}((z_0, z_1, z_2), (z_0, z_1, z_2)) = |z_1|^2 - 2\operatorname{Re}(\overline{z_0}z_2).$$

We will refer to a vector of norm 0 as a *null vector*.

Abusing notation, we will also use \mathbb{J} to denote the skew-diagonal matrix above. Note that \mathbb{J} has signature $(2, 1)$: it has two positive and one negative eigenvalue.

The unitary group $SU(2, 1) \subset GL(3, \mathbb{C})$ is the set of matrices $M \in SL(3, \mathbb{C})$ satisfying $\mathbb{J}(M\vec{z}, M\vec{w}) = \mathbb{J}(\vec{z}, \vec{w})$ for all $\vec{z}, \vec{w} \in \mathbb{C}^3$. Equivalently, M satisfies $M^\dagger \mathbb{J} M = \mathbb{J}$, where \dagger denotes the conjugate transpose. It is easy to see that:

Lemma 2.5. $U(\mathbb{H}) \subset SU(2, 1)$.

Later calculations will require us to step outside of $U(\mathbb{H})$. The following lemma provides a basic property of elements of $SU(2, 1)$.

Lemma 2.6. *Every matrix in $SU(2, 1)$ is of the form*

$$(2.4) \quad \begin{pmatrix} a_{1,1} & \overline{a_{2,3}a_{1,1}} - \overline{a_{2,1}a_{1,3}} & a_{1,3} \\ a_{2,1} & \overline{a_{3,3}a_{1,1}} - \overline{a_{3,1}a_{1,3}} & a_{2,3} \\ a_{3,1} & \overline{a_{3,3}a_{2,1}} - \overline{a_{3,1}a_{2,3}} & a_{3,3} \end{pmatrix}$$

Proof. Every matrix $M = (a_{i,j})$ in $SU(2, 1)$ satisfies $M^\dagger \mathbb{J} = \mathbb{J} M^{-1}$. We have

$$M^\dagger \mathbb{J} = \begin{pmatrix} -\overline{a_{3,1}} & \overline{a_{2,1}} & -\overline{a_{1,1}} \\ -\overline{a_{3,2}} & \overline{a_{2,2}} & -\overline{a_{1,2}} \\ -\overline{a_{3,3}} & \overline{a_{2,3}} & -\overline{a_{1,3}} \end{pmatrix}.$$

On the other hand,

$$\mathbb{J} M^{-1} = \begin{pmatrix} a_{3,1}a_{2,2} - a_{3,2}a_{2,1} & a_{3,2}a_{1,1} - a_{3,1}a_{1,2} & a_{2,1}a_{1,2} - a_{2,2}a_{1,1} \\ a_{3,1}a_{2,3} - a_{3,3}a_{2,1} & a_{3,3}a_{1,1} - a_{3,1}a_{1,3} & a_{2,1}a_{1,3} - a_{2,3}a_{1,1} \\ a_{3,2}a_{2,3} - a_{3,3}a_{2,2} & a_{3,3}a_{1,2} - a_{3,2}a_{1,3} & a_{2,2}a_{1,3} - a_{2,3}a_{1,2} \end{pmatrix}.$$

Comparing the two matrices completes the lemma. \square

2.4. Siegel Model. The Siegel model provides a geometric view of the unitary representation and a simpler formula for the Koranyi inversion. We will in fact define two closely related models, the *planar Siegel model* that views a point $h \in \mathbb{H}$ as a vector $(u, v) \in \mathbb{C}^2$, and the *projective Siegel model* that views h as a point in complex projective space with homogeneous coordinates $(1 : u : v)$. We will denote both models by \mathcal{S} .

We first identify a point $h \in \mathbb{H}$ with geometric coordinates (z, t) with the vector

$$(2.5) \quad (1, z(1 + \mathbf{i}), |z|^2 + \mathbf{i}t) \in \mathbb{C}^3.$$

Note that this is exactly the image of the vector $(1, 0, 0)$ under the unitary transformation $U(z, t)$. We will say that h has *planar Siegel coordinates*

$$(2.6) \quad (z(1 + \mathbf{i}), |z|^2 + \mathbf{i}t) \in \mathbb{C}^2.$$

The *planar Siegel model* of \mathbb{H} is the set of points in \mathbb{C}^2 of the form (2.6).

Some times, a unitary transformation will take $(1, z(1 + \mathbf{i}), |z|^2 + \mathbf{i}t)$ to a point that is not of the same form, but can be rescaled to be such. It will therefore be useful to think of vectors up to rescaling, that is, as elements of complex projective space \mathbb{CP}^2 .

Recall that the complex projective plane \mathbb{CP}^2 is the projectivization of \mathbb{C}^3 , i.e. the set of non-zero vectors up to rescaling by a non-zero complex number. A point in \mathbb{CP}^2 has *homogeneous coordinates* $(z_0 : z_1 : z_2)$, well-defined up to rescaling.

We can now define the *projective Siegel model* of \mathbb{H} as the set of points in \mathbb{CP}^2 with homogeneous coordinates $(1 : z(1 + \mathbf{i}) : |z|^2 + \mathbf{i}t)$.

Abusing notation, we will denote both Siegel models by \mathcal{S} , with the ideintification $(u, v) \leftrightarrow (1 : u : v)$. We have the following simple characterization of points in \mathcal{S} .

Lemma 2.7. *Let $(z_0 : z_1 : z_2) \in \mathbb{CP}^2$ be a null point, that is $\|(z_0, z_1, z_2)\|_{\mathbb{J}}^2 = 0$. Then either $(z_0 : z_1 : z_2) \in \mathcal{S}$ or $(z_0 : z_1 : z_2) \cong (0 : 0 : 1)$.*

Remark 2.8. The region $\{(z_0 : z_1 : z_2) \in \mathbb{CP}^2 : \|(z_0, z_1, z_2)\|_{\mathbb{J}}^2 < 0\}$ bounded by $\mathcal{S} \cup \{(0 : 0 : 1)\}$ is the *Siegel domain*. Complex hyperbolic space is defined on this region and has strong connections to the Heisenberg group, see e.g. [2, 5, 9, 10]. In particular, we intend to discuss the relation of Heisenberg continued fractions to geodesic coding in complex hyperbolic space in an upcoming paper, following [16].

Note that the gauge norm is easy to write in the Siegel model:

Lemma 2.9. *Let $(u, v) \in \mathcal{S}$. Then the gauge norm of (u, v) is $\|(u, v)\| = |v|^{1/2}$.*

Proof. An element of \mathcal{S} has the form $(u, v) = (z(1 + \mathbf{i}), |z|^2 + \mathbf{i}t)$ for some $(z, t) \in \mathbb{H}$. The gauge norm of (z, t) is given by $\|(z, t)\| = \sqrt[4]{|z|^4 + t^2} = |v|^{1/2}$. \square

The gauge distance is defined as $d(h, k) = \|h^{-1}k\|$. With this in mind, we show:

Lemma 2.10. *In the planar Siegel model, we have*

$$(u_1, v_1)^{-1}(u_2, v_2) = (u_2 - u_1, \overline{v_1} - \overline{u_1}u_2 + v_2).$$

Proof. We have associated to $(u_1, v_1)^{-1}$ the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -u_1 & 1 & 0 \\ \overline{v_1} & -\overline{u_1} & 1 \end{pmatrix}$$

Applying this matrix to the point $(1, u_2, v_2)$, we get the vector

$$(1, u_2 - u_1, \overline{v_1} - \overline{u_1}u_2 + v_2).$$

Taking the last two coordinates yields the desired formula. \square

We now study the action of $SU(2, 1)$ matrices on the Heisenberg group in the Siegel models. General linear matrices act on \mathbb{CP}^2 by acting on the homogeneous coordinates. Since we have $\mathbb{C}^2 \hookrightarrow \mathbb{CP}^2$ by taking $(u, v) \mapsto (1 : u : v)$, we also obtain an action on \mathbb{C}^2 .

Lemma 2.11. *Let $M = (a_{i,j}) \in GL(3, \mathbb{C})$ and $(u, v) \in \mathbb{C}^2 \hookrightarrow \mathbb{CP}^2$. Then M acts on (u, v) as:*

$$M(u, v) = \left(\frac{a_{2,1} + a_{2,2}u + a_{2,3}v}{a_{1,1} + a_{1,2}u + a_{1,3}v}, \frac{a_{3,1} + a_{3,2}u + a_{3,3}v}{a_{1,1} + a_{1,2}u + a_{1,3}v} \right).$$

Proof. The point (u, v) corresponds to a point in \mathbb{CP}^2 with homogeneous coordinates $(1 : u : v)$. We then have

$$M \begin{pmatrix} 1 \\ u \\ v \end{pmatrix} = \begin{pmatrix} a_{1,1} + a_{1,2}u + a_{1,3}v \\ a_{2,1} + a_{2,2}u + a_{2,3}v \\ a_{3,1} + a_{3,2}u + a_{3,3}v \end{pmatrix}$$

To view $M(1 : u : v)$ as a point in \mathbb{C}^2 , we renormalize so that the first coordinate is 1, and take the remaining two coordinates. \square

Elements of $GL(3, \mathbb{C})$ do not necessarily preserve the set \mathcal{S} , but the unitary matrices $SU(2, 1)$ preserve \mathbb{J} and therefore \mathcal{S} . In particular, elements of $\mathbb{U}(\mathbb{H})$ act transitively on \mathcal{S} while fixing the point $(0 : 0 : 1)$. Denote the matrix $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ by $\mathbb{U}(\iota)$.

Lemma 2.12. *$\mathbb{U}(\iota)$ acts on \mathbb{H} by the Koranyi inversion ι .*

Proof. We compute, for a point in \mathbb{H} with geometric coordinates (z, t) and projective Siegel coordinates $(1 : z(1 + \mathbf{i}) : |z|^2 + t\mathbf{i})$:

$$\begin{aligned} \mathbb{U}(\iota)(1 : z(1 + \mathbf{i}) : |z|^2 + t\mathbf{i}) &= (|z|^2 + t\mathbf{i} : -z(1 + \mathbf{i}) : 1) \\ &= \left(1 : \frac{-z}{|z|^2 + t\mathbf{i}}(1 + \mathbf{i}) : \frac{1}{|z|^2 + t\mathbf{i}} \right) \\ &= \left(1 : \frac{-z}{|z|^2 + t\mathbf{i}}(1 + \mathbf{i}) : \frac{|z|^2 - t\mathbf{i}}{|z|^4 + t^2} \right) \\ &= \left(1 : \frac{-z}{|z|^2 + t\mathbf{i}}(1 + \mathbf{i}) : \left| \frac{-z}{|z|^2 + t\mathbf{i}} \right|^2 + \frac{-t}{|z|^4 + t^2}\mathbf{i} \right) \end{aligned}$$

We thus have that under $\mathbb{U}(\iota)$, the geometric coordinates (z, t) are mapped to $\left(\frac{-z}{|z|^2 + t\mathbf{i}}, \frac{-t}{\| (z, t) \|} \right)$, as desired. \square

2.5. Lattices and Fundamental Domains. Recall that $\mathbb{H}(\mathbb{Z})$ is the set of Heisenberg points with integer coordinates. In the geometric model $\mathbb{H} = \mathbb{C} \times \mathbb{R}$, we have $\mathbb{H}(\mathbb{Z}) = \mathbb{Z}[\mathbf{i}] \times \mathbb{Z}$. In the Siegel model, $\mathbb{H}(\mathbb{Z})$ is the set of points (u, v) such that $u, v \in \mathbb{Z}[\mathbf{i}]$, and u has norm divisible by 2. In the unitary model, we have $\mathbb{H}(\mathbb{Z}) \subset SU(2, 1; \mathbb{Z}[\mathbf{i}])$, where the latter denotes the set of unitary matrices with Gaussian integer coefficients, and is known as the Picard modular group.

Likewise, we will denote by $\mathbb{H}(\mathbb{Q})$ the set of points in \mathbb{H} with rational coordinates. Recall that the Heisenberg group admits a family of dilation maps δ_r given by

$\delta_r(z, t) = (rz, r^2t)$ in the geometric model. The dilation maps are group isomorphisms and satisfy $d(\delta_r h, \delta_r q) = r \cdot d(h, q)$ for all $h, q \in \mathbb{H}$ and $r \geq 0$. It is clear that $h \in \mathbb{H}(\mathbb{Q})$ if and only if there is an integer $n \in \mathbb{N}$ such that $\delta_n h \in \mathbb{H}(\mathbb{Z})$.

We are now interested in the structure and geometry of $\mathbb{H}(\mathbb{Z})$. We record its generators in the geometric model:

Lemma 2.13. *The group $\mathbb{H}(\mathbb{Z})$ is generated by the elements $(1, 0)$, $(i, 0)$, and $(0, 1)$.*

As Falbel–Francis–Lax–Parker showed, $\mathbb{Z}[i]$ and $SU(2, 1; \mathbb{Z}[i])$ are closely linked:

Theorem 2.14 ([4]). *The group $SU(2, 1; \mathbb{Z}[i])$ is generated by the matrices $\mathbb{U}(1, 0)$, $\mathbb{U}(0, 1)$, $\mathbb{U}(\iota)$, and iI , where I is the identity matrix (see §2.3, 2.4 for the notation).*

Remark 2.15. Note that we defined $SU(2, 1; \mathbb{Z}[i])$ with a particular Hermitian form \mathbb{J} in mind. Different Hermitian forms \mathbb{J} provide isomorphic Lie groups $U(2, 1)$, but the lattice $SU(2, 1; \mathbb{Z}[i])$ depends on the choice of the Hermitian form. If two forms are related by an integer change of coordinates, then the associated lattices are equivalent. If the change of coordinates is not integral, the lattices are not isomorphic as groups (even up to finite index), see [13, 14]. Nonetheless, in literature one mostly sees mention of the Picard modular group, defined by a Hermitian form equivalent to our \mathbb{J} .

We now discuss fundamental domains for $\mathbb{H}(\mathbb{Z})$. Recall that a fundamental domain for $\mathbb{H}(\mathbb{Z})$ is a connected set $K \subset \mathbb{H}$ with piecewise smooth boundary whose translates tile \mathbb{H} without overlap. That is, $\cup\{g*K : g \in \mathbb{H}(\mathbb{Z})\} = \mathbb{H}$ and $K \cap (g*K) \neq \emptyset$ implies $g = 0$.

Our definition of fundamental domain is slightly stronger than usual. The standard definition only considers $\overset{\circ}{K} \cap (g*\overset{\circ}{K}) \neq \emptyset$, where $\overset{\circ}{K}$ is the interior of K . We require “half-closed” fundamental domains, so that the following holds:

Lemma 2.16. *Let K be a fundamental domain for $\mathbb{H}(\mathbb{Z})$. Then the map $[p]_K$ mapping all points of gK to g is well-defined.*

The following lemma follows immediately from the definitions:

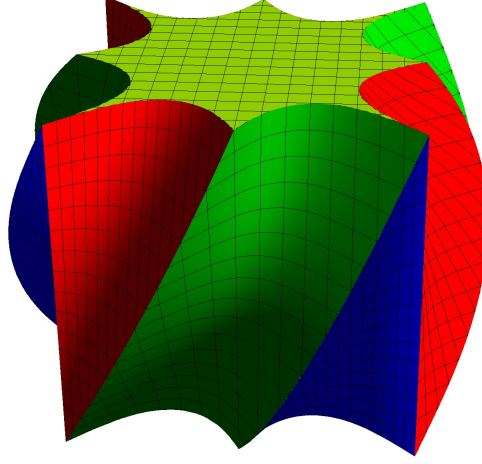
Lemma 2.17. *The following regions are fundamental domains for $\mathbb{H}(\mathbb{Z})$:*

- *The unit cube $K_C = [-1/2, 1/2) \times [-1/2, 1/2) \times [-1/2, 1/2)$.*
- *The Dirichlet domain $K_D = \{h \in \mathbb{H} : d(0, h) \leq d(g, h) \text{ for all } g \in \mathbb{H}(\mathbb{Z})\}$, with a choice of excluded boundary points.*

Denote the unit sphere in \mathbb{H} by S . For a subset $A \subset \mathbb{H}$, let $\text{rad}(A)$ denote the supremum of the norms of the points of A , and let $\lambda(A)$ denote its Lebesgue measure (in the geometric model).

Lemma 2.18. *Every fundamental domain K for $\mathbb{H}(\mathbb{Z})$ satisfies $\lambda(K) = 1$. Furthermore, the domains K_C and K_D satisfy $\text{rad}(K_C) = \text{rad}(K_D) = \sqrt[4]{1/2}$.*

Proof. The radius of K_C is easy to compute because $\|\cdot\|$ behaves similarly to the Euclidean norm. As in the Euclidean case, the norm is maximized by each corner of the cube. We have $\|(1/2 + i1/2, 1/2)\| = \sqrt[4]{1/2}$.

FIGURE 3. The Dirichlet domain for $\mathbb{H}(\mathbb{Z})$ centered at the origin.

The radius of K_D seems difficult to compute directly, as the boundary of K_D is more complicated (see Figure 3). We will therefore argue indirectly by means of K_C . Let $h \in K_D$, and choose $g \in \mathbb{H}(\mathbb{Z})$ so that $g * h \in K_C$. We then have $\|g * h\| \leq \text{rad}(K_C) = \sqrt[4]{1/2}$. This implies that $d(g^{-1}, h) \leq \sqrt[4]{1/2}$. Now, by definition of K_D , $d(0, h) \leq d(g^{-1}, h) \leq \sqrt[4]{1/2}$, so we must also have $\|h\| \leq \sqrt[4]{1/2}$. To prove equality, one shows that the point $(1/2 + i1/2, 1/2)$ is contained in K_D .

For the volume computation, it is clear that $\lambda(K_C) = 1$. To compute $\lambda(K)$ for an arbitrary fundamental domain K , note that Lebesgue measure is preserved by left translation in the Heisenberg group (which acts by shears). Since K_C can be constructed by rearranging measurable pieces of K , the two fundamental domains must have the same volume. \square

3. HEISENBERG CONTINUED FRACTIONS

Fix a fundamental domain K for the group $\mathbb{H}(\mathbb{Z})$ such that $\text{rad}(K) < 1$ (e.g. K_C or K_D in Lemma 2.17). We begin by establishing some notation.

Definition 3.1. Given an arbitrary sequence $\{\gamma_i\}_{i=1}^n$ of non-zero digits in $\mathbb{H}(\mathbb{Z})$, we write the associated continued fraction as

$$(3.1) \quad \mathbb{K}\{\gamma_i\} = \mathbb{K}\{\gamma_i\}_{i=1}^n = \iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_n.$$

For an infinite sequence $\{\gamma_i\}_{i=1}^\infty$, we define $\mathbb{K}\{\gamma_i\} = \mathbb{K}\{\gamma_i\}_{i=1}^\infty := \lim_{n \rightarrow \infty} \mathbb{K}\{\gamma_i\}_{i=1}^n$, if this limit exists. The goal of this section is to show that the limit does exist in several important cases, and that the computation of $\mathbb{K}\{\gamma_i\}$ may be simplified by using a recursive algorithm.

Definition 3.2. We associate with K :

- (1) A “nearest-integer” map $[\cdot] : \mathbb{H} \rightarrow \mathbb{H}(\mathbb{Z})$, characterized by

$$[h] = g \text{ for each } g \in \mathbb{H}(\mathbb{Z}) \text{ and } h \in gK.$$

Note that $[\cdot]$ selects the nearest integer in the gauge metric exactly if K is the Dirichlet domain K_D .

- (2) The *Gauss map* $T : K \setminus \{0\} \rightarrow K$ given by

$$T(h) = [\iota h]^{-1} \iota h.$$

Remark 3.3. Working with the geometric model, one sees that each axis is preserved by the Gauss map T . In particular, the restriction of T to each axis is essentially isomorphic to the nearest-integer Gauss map on $[-1/2, 1/2]$. The theory of continued fractions we develop likewise restricts to the classical nearest-integer continued fraction theory on the axes.

Definition 3.4. Given a point $h \in K$, have:

- (1) The *forward iterates* $h_i := T^i h \in K$, for each i ,
- (2) The *continued fraction digits* $\gamma_i := [\iota h_{i-1}] \in \mathbb{H}(\mathbb{Z})$, for each i ,
- (3) The *rational approximants* $\mathbb{K}\{\gamma_i\}_{i=1}^n \in \mathbb{H}(\mathbb{Q})$, for each n .

Because T is defined on $K \setminus \{0\}$, the process of defining forward iterates, continued fraction digits, and rational approximants terminates if for some i we have $h_i = 0$. We will characterize the points h for which this happens in Theorem 3.10.

More generally, for a point $h \in \mathbb{H}$ we can take $\gamma_0 = [h]$, $h_0 = [\gamma_0^{-1}h]$ and obtain the remaining digits $\{\gamma_i\}_{i=1}^\infty$ of $CF(h)$ from $h_0 \in K$ as before. However, our focus will be on points in K .

It is easy to see that, on finite sequences, \mathbb{K} is the inverse operation to CF :

Lemma 3.5. *For $h \in K$ with $CF(h)$ a finite sequence, we have $\mathbb{K}CF(h) = h$.*

Remark 3.6. The operation \mathbb{K} is defined without reference to a specific fundamental domain K . Thus, while we will show that $\mathbb{K}CF(h) = h$, we do not in general have $CF(\mathbb{K}\{\gamma_i\}) = \{\gamma_i\}$. Indeed, problems arise when the γ_i get too close to the unit sphere.

For example, let $K = K_C$, the unit cube, and $\{\gamma_i\} = \{(a_1, b_1) = (1, 0)\}$. We have

$$\mathbb{K}\{\gamma_i\} = \iota(1, 0) = (-1, 0).$$

Attempting to reverse the process, we have $(a_0, b_0) = [(-1, 0)] = (-1, 0)$, and $(-1, 0)^{-1} * (-1, 0) = (0, 0)$, so that

$$CF(-1, 0) = \{(a_0, b_0) = (-1, 0)\}.$$

This non-uniqueness of continued fraction expansions is analogous to how in regular continued fractions we have, for example,

$$\frac{1}{5 + \frac{1}{1}} = \frac{1}{6}.$$

3.1. Pringsheim-Type Theorem. The Pringsheim Theorem for regular continued fractions guarantees convergence of a continued fraction whose digits are sufficiently large. A variant holds for the Heisenberg group:

Theorem 3.7 (Pringsheim-Type Theorem). *Let $\{\gamma_i\}_{i=1}^\infty$ be a sequence of points in $\mathbb{H}(\mathbb{Z})$ such that for each i we have $\|\gamma_i\| \geq 3$. Then the limit $\mathbb{K}\{\gamma_i\}$ exists. Furthermore, $CF(\mathbb{K}\{\gamma_i\}) = \{\gamma_i\}$.*

Proof. Recall that left multiplication by any $\gamma \in \mathbb{H}(\mathbb{Z})$ is an isometry, and ι satisfies, for all $h, k \in \mathbb{H}$, $d(\iota h, \iota k) = \frac{d(h, k)}{\|h\|\|k\|}$ (Lemma 2.12).

Let K_D be the Dirichlet fundamental domain for $\mathbb{H}(\mathbb{Z})$. It follows from the definition of K_D and the triangle inequality that for each point $h \in \mathbb{H}$ with $\|h\| < 1/2$, we have $h \in K_D$. Conversely, for each point $h \in K_D$ we have by Lemma 2.18 that $\|h\| \leq \sqrt[4]{1/2}$.

Suppose that $\gamma \in \mathbb{H}(\mathbb{Z})$ with $\|\gamma\| \geq 3$. We claim that $\iota\gamma K_D \subset K_D$. Indeed, every point $h \in \gamma K_D$ satisfies $\|\gamma h\| \geq 3 - \|h\| \geq 3 - \sqrt[4]{1/2} > 2$, so that $\|\iota\gamma h\| < \frac{1}{2}$, and we conclude $\iota\gamma K_D \subset K_D$.

Now, for each n , we have (because the identity element 0 is contained in K_D):

$$\begin{aligned} \mathbb{K}\{\gamma_i\}_{i=1}^n &= \iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_n \\ &= \iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_n 0 \\ &\in \iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_n K_D. \end{aligned}$$

These *cylinder sets* form a nested sequence:

$$\begin{aligned} \iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_n K_D &= \iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_{n-1}(\iota\gamma_n K_D) \\ &\subset \iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_{n-1} K_D. \end{aligned}$$

By the above calculation, the diameter of the cylinder set $\iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_n K_D$ is bounded above by $(3 - \sqrt[4]{1/2})^{-2n} \text{diam}(K_D)$. We thus have that the sequence of fractions $\mathbb{K}\{\gamma_i\}_{i=1}^n$ (as n varies) is a Cauchy sequence, and hence converges to some $\mathbb{K}\{\gamma_i\}$.

We thus have that $\mathbb{K}\{\gamma_i\}$ exists. By construction, we also know that it is contained in the cylinder sets $\iota\gamma_1\iota\gamma_2 \cdots \iota\gamma_n K_D$, for each n (note that the cylinder sets are in fact properly nested, so that $\mathbb{K}\{\gamma_i\}$ cannot escape to a cylinder set's boundary). This is equivalent to the second assertion of the theorem. \square

3.2. Rational Points. We will now show that a point in \mathbb{H} has rational coordinates if and only if it has a finite continued fraction expansion. Our proof is motivated by the work of Falbel–Francsics–Lax–Parker [4] and uses the Siegel model.

Recall that for a point $h \in K$ that is of interest to us, we will write

$$h = (u, v) \in \mathbb{C}^2$$

in the planar Siegel model. We will also think of (u, v) as the element of \mathbb{CP}^2 with homogeneous coordinates $(1 : u : v)$. That is, it is the vector $(1, u, v)$ considered up to multiplication by a non-zero complex number.

Definition 3.8. Given an element $\gamma \in \mathbb{H}(\mathbb{Z})$ with planar Siegel coordinates $(\alpha, \beta) \in (\mathbb{Z}[\mathbf{i}] \times \mathbb{Z}[\mathbf{i}]) \cap \mathcal{S}$, define

$$\begin{aligned} A_\gamma &:= \mathbb{U}(\iota)\mathbb{U}(\gamma) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \bar{\alpha} & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\beta & -\bar{\alpha} & -1 \\ \alpha & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Lemma 3.9. *In the Siegel projective model, we have*

$$\mathbb{K}\{\gamma_i\}_{i=1}^n = A_{\gamma_1} \cdots A_{\gamma_n} (1 : 0 : 0).$$

Proof. Abstractly, we have the definition $\mathbb{K}\{\gamma_i\}_{i=1}^n = \iota\gamma_1\iota \cdots \iota\gamma_n$. Using the identity element $0 \in \mathbb{H}$, we may also write $\mathbb{K}\{\gamma_i\}_{i=1}^n = \iota\gamma_1\iota \cdots \iota\gamma_n 0$. In the projective Siegel model, 0 is interpreted as the point $(1 : 0 : 0) \in \mathbb{CP}^2$. The inversion ι and left multiplication by γ_i are, respectively, interpreted as the unitary matrices $\mathbb{U}(\iota)$ and $\mathbb{U}(\gamma_i)$. Thus, $\mathbb{K}\{\gamma_i\}_{i=1}^n = A_{\gamma_1} \cdots A_{\gamma_n} (1 : 0 : 0)$, as desired. \square

We are now in position to characterize rational Heisenberg points in terms of their continued fraction expansion.

Theorem 3.10. *Let $h \in \mathbb{H}$. Then $h \in \mathbb{H}(\mathbb{Q})$ if and only if $h = \mathbb{K}\{\gamma_i\}_{i=0}^n$ for some finite sequence $\{\gamma_i\}_{i=0}^n$.*

Proof. Suppose $h = \mathbb{K}\{\gamma_i\}_{i=0}^n$. Then it is clear from the definitions of \mathbb{K} and the fact that $\gamma_i \in \mathbb{H}(\mathbb{Z})$ that $h \in \mathbb{H}(\mathbb{Q})$.

Conversely, fix $K = K_D$ and assume by way of contradiction that there exists an element $h \in \mathbb{H}(\mathbb{Q})$ with an infinite continued fraction sequence $CF(h) = \{\gamma_i\}_{i=1}^\infty$. Without loss of generality, we may assume $h \in K$ (this corresponds to discarding the digit γ_0 of h).

The idea of the proof is to show that the forward iterates h_i of h can be written as fractions whose denominators decrease with i . Write, in planar Siegel coordinates,

$$h = \left(\frac{r}{q}, \frac{p}{q} \right),$$

with $q, r, p \in \mathbb{Z}[\mathbf{i}]$. Because $h \in K$, we have by Lemma 2.9 that $|p/q| \leq \text{rad}(K)^2 < 1$.

Consider the first forward iterate $h_1 = Th = \gamma_1^{-1}\iota h$ as a vector in \mathbb{C}^3 :

$$\begin{pmatrix} q^{(1)} \\ r^{(1)} \\ p^{(1)} \end{pmatrix} := A_{\gamma_1}^{-1} \begin{pmatrix} q \\ r \\ p \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & \alpha_1 \\ -1 & -\bar{\alpha}_1 & -\bar{\beta}_1 \end{pmatrix} \begin{pmatrix} q \\ r \\ p \end{pmatrix} = \begin{pmatrix} -p \\ r + \alpha_1 p \\ -q - \bar{\alpha}_1 r - \bar{\beta}_1 p \end{pmatrix}$$

Thus, h_1 is a rational point with planar Siegel coordinates $h_1 = \left(\frac{r^{(1)}}{q^{(1)}}, \frac{p^{(1)}}{q^{(1)}} \right)$. Furthermore, we have $q^{(1)} = p$, so that

$$(3.2) \quad \left| \frac{q^{(1)}}{q} \right| = \left| \frac{p}{q} \right| = \|h\|^2 < \text{rad}(K) < 1.$$

Repeating this procedure recursively, we have rational coordinates $h_i = \left(\frac{r^{(i)}}{q^{(i)}}, \frac{p^{(i)}}{q^{(i)}}\right)$ for each forward iterate h_i , satisfying $q^{(i)} = p^{(i-1)}$. Since $h_i \in K$ for each i , we obtain for each n :

$$(3.3) \quad |q^{(n)}| \leq |q| (\text{rad}(K))^{2n}$$

For sufficiently large n , we conclude $|q_n| < 1$, which implies that $q_n = 0$, but that is only possible if $h_{n-1} = 0$ and $CF(h)$ is, in fact, finite. \square

As a corollary to the proof of Theorem 3.10, we obtain

Theorem 3.11 (Denominator Growth Theorem). *Let $h \in \mathbb{H}(\mathbb{Q})$, with $CF(h) = \{\gamma_i\}_{i=0}^n$. Suppose one can write h as a fraction with denominator $q \in \mathbb{Z}[\mathfrak{i}]$. Then,*

$$|q| \geq 2^{n/2}.$$

Proof. The result follows directly from (3.3), using either fundamental domain K in Lemma 2.18, with radius bounded by $\sqrt[4]{1/2}$, and the fact that $q_n \neq 0$. \square

Remark 3.12. One may hope for a stronger statement that for a sequence $\{\gamma_i\}_{i=1}^\infty$ of elements of $\mathbb{H}(\mathbb{Z})$, the norms of the denominators q_n of the partial fractions $\mathbb{K}\{\gamma_i\}_{i=1}^n$ are an increasing sequence. However, we are unable to prove this without assuming that $\|\gamma_i\| \geq 2$ for all i .

3.3. Recursive Formula. We will now find a simple recursive formula for $\mathbb{K}\{\gamma_i\}_{i=1}^\infty$.

Definition 3.13. Let $\{\gamma_i\}$ be a sequence of elements of $\mathbb{H}[\mathbb{Z}]$. Define

$$Q_n := A_{\gamma_1} \cdots A_{\gamma_n},$$

$$(q_n, r_n, p_n) := Q_n(1, 0, 0).$$

We have the following by Lemma 3.9.

Lemma 3.14. *In the above notation, $\mathbb{K}\{\gamma_i\}_{i=1}^n = \left(\frac{r_n}{q_n}, \frac{p_n}{q_n}\right)$, in the planar Siegel model.*

Remark 3.15. It should be noted that Theorem 3.11 does not imply that $q_n \geq 2^{n/2}$. Recall from Remark 3.6 that if $CF(h) = \{\gamma_i\}_{i=0}^\infty$, we do not necessarily have that $CF(\mathbb{K}\{\gamma_i\}_{i=0}^n) = \{\gamma_i\}_{i=0}^n$.

Lemma 3.14 states that the partial fraction $\mathbb{K}\{\gamma_i\}_{i=1}^n$ is encoded in the matrix Q_n . As in the case of regular continued fractions, Q_n stores additional information:

Lemma 3.16. *In the above notation, the matrices Q_n have the form*

$$Q_n = \begin{pmatrix} q_n & \tilde{q}_n & -q_{n-1} \\ r_n & \tilde{r}_n & -r_{n-1} \\ p_n & \tilde{p}_n & -p_{n-1} \end{pmatrix},$$

where the elements $\tilde{q}_n, \tilde{r}_n, \tilde{p}_n$ are given by:

$$\begin{aligned} \tilde{q}_n &= \overline{r_n q_{n-1} - q_n r_{n-1}}, \\ \tilde{r}_n &= \overline{p_n q_{n-1} - q_n p_{n-1}}, \\ \tilde{p}_n &= \overline{p_n r_{n-1} - r_n p_{n-1}}. \end{aligned}$$

Proof. The first column of Q_n is as stated by the definition of the vector (q_n, r_n, p_n) . The third column follows from the identity $Q_n = Q_{n-1}A_{\gamma_n}$. The second column follows from the “cross product” Lemma 2.6. \square

We record the following for later use:

Lemma 3.17. *The following formula holds for Q_n :*

$$\begin{aligned} & \overline{\begin{pmatrix} -p_n & r_n & -q_n \\ -\tilde{p}_n & \tilde{r}_n & -\tilde{q}_n \\ -p_{n-1} & r_{n-1} & -q_{n-1} \end{pmatrix}} \\ &= \begin{pmatrix} p_n \tilde{r}_n - \tilde{p}_n r_n & \tilde{p}_n q_n - \tilde{q}_n p_n & r_n \tilde{q}_n - \tilde{r}_n q_n \\ p_{n-1} r_n - p_n r_{n-1} & p_n q_{n-1} - p_{n-1} q_n & r_{n-1} q_n - r_n q_{n-1} \\ p_{n-1} \tilde{r}_n - \tilde{p}_n r_{n-1} & \tilde{p}_n q_{n-1} - p_{n-1} \tilde{q}_n & r_{n-1} \tilde{q}_n - \tilde{r}_n q_{n-1} \end{pmatrix}. \end{aligned}$$

Proof. This follows from the identity $Q_n^\dagger \mathbb{J} = \mathbb{J} Q_n^{-1}$ and Lemma 3.16. \square

We can now obtain a recursive form for the partial fractions $\mathbb{K}\{\gamma_i\}_{i=1}^n$.

Theorem 3.18. *Let $\{\gamma_i\}_{i=1}^\infty$ be a sequence of elements of $\mathbb{H}(\mathbb{Z})$ represented in the planar Siegel model by the vectors $\{(\alpha_i, \beta_i)\}_{i=1}^\infty$. Let $(q_{-1}, p_{-1}, r_{-1}) = (0, 0, 1)$ and $(q_0, p_0, r_0) = (1, 0, 0)$. Define, recursively, for $n \geq 0$,*

$$\begin{pmatrix} q_{n+1} \\ r_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} q_n & \overline{r_n q_{n-1} - q_n r_{n-1}} & -q_{n-1} \\ r_n & \overline{p_n q_{n-1} - q_n p_{n-1}} & -r_{n-1} \\ p_n & \overline{p_n r_{n-1} - r_n p_{n-1}} & -p_{n-1} \end{pmatrix} \begin{pmatrix} -\beta_{n+1} \\ \alpha_{n+1} \\ -1 \end{pmatrix}$$

Then for each n we have, in the planar Siegel model,

$$\mathbb{K}\{\gamma_i\}_{i=1}^n = \left(\frac{r_n}{q_n}, \frac{p_n}{q_n} \right).$$

Proof. Earlier in the section, we defined matrices A_{γ_i} (which append the digit γ_i to a continued fraction) and $Q_n = A_{\gamma_1} \cdots A_{\gamma_n}$. We set $(q_n, r_n, p_n) = Q_n(1, 0, 0)$. We claim that this agrees with the definition in the statement of the theorem. Lemma 3.14 will then tell us that $\mathbb{K}\{\gamma_i\}_{i=1}^n = \left(\frac{r_n}{q_n}, \frac{p_n}{q_n} \right)$.

Taking Q_0 to be the identity matrix, the following computation provides the equivalence (see the definition of $A_{\gamma_{n+1}}$ and Lemma 3.16 for the form of the two matrices).

$$\begin{aligned}
\begin{pmatrix} q_{n+1} \\ r_{n+1} \\ p_{n+1} \end{pmatrix} &= Q_{n+1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= Q_n A_{\gamma_{n+1}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} q_n & \tilde{q}_n & -q_{n-1} \\ r_n & \tilde{p}_n & -r_{n-1} \\ p_n & \tilde{r}_n & -p_{n-1} \end{pmatrix} \begin{pmatrix} -\beta_{n+1} & -\bar{\alpha}_{n+1} & -1 \\ \alpha_{n+1} & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} q_n & \tilde{q}_n & -q_{n-1} \\ r_n & \tilde{p}_n & -r_{n-1} \\ p_n & \tilde{r}_n & -p_{n-1} \end{pmatrix} \begin{pmatrix} -\beta_{n+1} \\ \alpha_{n+1} \\ -1 \end{pmatrix}
\end{aligned}$$

Rewriting $\tilde{q}_n, \tilde{r}_n, \tilde{p}_n$ in terms of the other terms in Q_n completes the proof. \square

3.4. Continued Fraction Representation Theorem. We are now ready to prove the convergence of continued fraction expansions. In fact, we obtain a variation on the strong convergence property. While we do not obtain strong convergence in the sense of Schweiger [15], our convergence estimate is obtained via a similar method to strong convergence for regular continued fractions. We hope to improve this estimate and explore applications to Diophantine approximation in an upcoming paper.

We also note that we obtain such an explicit convergence estimate by exploiting a special form for Q_n^{-1} that follows from the identity $M^\dagger \mathbb{J} M = \mathbb{J}$ that defines $U(2, 1)$. Other continued fraction theories are complicated by the lack of a simple form for Q_n^{-1} .

Before we can prove convergence, we need to show that q_n will never equal 0. We prove this in two steps.

Lemma 3.19. *We have*

$$(3.4) \quad \begin{pmatrix} q_n + \tilde{q}_n u_n - q_{n-1} v_n \\ r_n + \tilde{r}_n u_n - r_{n-1} v_n \\ p_n + \tilde{p}_n u_n - p_{n-1} v_n \end{pmatrix} = (-1)^n \begin{pmatrix} \frac{1}{v v_1 \cdots v_{n-1}} \\ u \\ \frac{1}{v v_1 \cdots v_{n-1}} \\ \frac{1}{v_1 \cdots v_{n-2}} \end{pmatrix}$$

Proof. By Lemma 3.16, the vector on the left-hand side of (3.4) equals

$$Q_n(1, u_n, v_n) = A_{\gamma_1} \cdots A_{\gamma_n}(1, u_n, v_n).$$

Recall that the forward iterates of h are given by $h_i = T^i h = A_{\gamma_i}^{-1} \cdots A_{\gamma_1}^{-1} h$, and have planar Siegel coordinates (u_i, v_i) , corresponding to the points $(1 : u_i : v_i) \in \mathbb{CP}^2$.

More generally, we have $A_{\gamma_i} \cdots A_{\gamma_n}(1 : u_n : v_n) = h_i$. Write $A_{\gamma_n}(1, u_n, v_n) =: (a, b, c)$. Since A_{γ_n} has the form (see Definition 3.8)

$$\begin{pmatrix} -\beta & -\bar{\alpha} & -1 \\ \alpha & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

we have that $c = -1$. Since $(b/a, c/a) = (u_{n-1}, v_{n-1})$, we conclude

$$A_{\gamma_n}(1, u_n, v_n) = \left(-\frac{1}{v_{n-1}}, -\frac{u_{n-1}}{v_{n-1}}, -1 \right).$$

Continuing in the same fashion we see that

$$\begin{aligned} A_{\gamma_{n-1}} A_{\gamma_n}(1, u_n, v_n) &= A_{\gamma_{n-1}} \left(-\frac{1}{v_{n-1}}, -\frac{u_{n-1}}{v_{n-1}}, -1 \right) \\ &= \left(\frac{1}{v_{n-1}v_{n-2}}, \frac{u_{n-2}}{v_{n-1}v_{n-2}}, \frac{1}{v_{n-1}} \right). \end{aligned}$$

After n iterations, the process yields the desired formula. \square

Lemma 3.20. *For $n \geq 0$, we have that q_n never equals 0.*

Proof. Assume, by way of contradiction, that $q_n = 0$. Then by Lemmas 3.16 and 3.17, we have that $\tilde{q}_n = 0$ as well (r_n also must equal 0, but we will not use this fact). Since the matrix Q_n has determinant $(-1)^n$ and each entry is a Gaussian integer, we must have $-q_{n-1}$ must have norm 1.

Therefore, we have that

$$(3.5) \quad |q_n + \tilde{q}_n u_n - q_{n-1} v_n| = |v_n| < 1$$

However by Lemma 3.19, we have that

$$(3.6) \quad |q_n + \tilde{q}_n u_n - q_{n-1} v_n| = |v v_1 v_2 \dots v_{n-1}|^{-1} > 1,$$

which is a contradiction. Therefore our assumption that $q_n = 0$ must be false. \square

Now we can continue with the proof of convergence.

Theorem 3.21. *Let $h \in \mathbb{H}$ and let K be a fundamental domain for $\mathbb{H}(\mathbb{Z})$ with $\text{rad}(K) < 1$. Then*

$$\mathbb{K}CF(h) = h.$$

Furthermore, if $CF(h) = \{\gamma_i\}$ is a sequence with at least n terms, then the rational approximants satisfy

$$d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) \leq \text{rad}(K)^{n+1}$$

for both rational and irrational points in \mathbb{H} . Let q_n be the denominator of the n^{th} rational approximate. Then we in fact have

$$d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) \leq \frac{\text{rad}(K)^{n+1}}{|q_n|^{1/2}}.$$

Proof. Recall from Lemma 3.14 that the associated rational approximates $\mathbb{K}\{\gamma_i\}_{i=1}^n$ have planar Siegel coordinates $\left(\frac{r_n}{q_n}, \frac{p_n}{q_n}\right)$, associated to the vector $(q_n, r_n, p_n) \in \mathbb{C}^3$. Recall also that the forward iterates $T^n h$ have planar Siegel coordinates (u_n, v_n) , and we have $|v_n|^{1/2} \leq \text{rad}(K) < 1$.

It therefore suffices to show that

$$d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) = \frac{\prod_{i=0}^n |v_i|^{1/2}}{|q_n|^{1/2}}.$$

Indeed, by Lemmas 2.10 and 2.9, we have

$$\begin{aligned} d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) &= d\left(\left(\frac{r_n}{q_n}, \frac{p_n}{q_n}\right), h\right) \\ &= \left\| \left(u - \frac{r_n}{q_n}, v - \overline{\left(\frac{r_n}{q_n}\right)}u + \overline{\left(\frac{p_n}{q_n}\right)}\right) \right\| \\ &= \left| v - \overline{\left(\frac{r_n}{q_n}\right)}u + \overline{\left(\frac{p_n}{q_n}\right)} \right|^{1/2} \\ &= \frac{|\overline{q_n}v - \overline{r_n}u + \overline{p_n}|^{1/2}}{|q_n|^{1/2}}. \end{aligned}$$

We now view h as the vector $(1, u, v)$ and represent the operation T^n by the unitary matrix Q_n^{-1} . The vector

$$Q_n^{-1} \begin{pmatrix} 1 \\ u \\ v \end{pmatrix} = \begin{pmatrix} -\overline{p_{n-1}} & \overline{r_{n-1}} & -\overline{q_{n-1}} \\ -\tilde{p}_n & \tilde{r}_n & -\tilde{q}_n \\ p_n & -r_n & q_n \end{pmatrix} \begin{pmatrix} 1 \\ u \\ v \end{pmatrix}$$

is then a scalar multiple of $(1, u_n, v_n)$. In particular,

$$v_n = -\frac{\overline{p_n} - \overline{r_n}u + \overline{q_n}v}{\overline{p_{n-1}} - \overline{r_{n-1}}u + \overline{q_{n-1}}v}.$$

By multiplying this formula together for various indices we obtain

$$\begin{aligned} \prod_{i=1}^n v_i &= (-1)^n \prod_{i=1}^n \frac{\overline{p_i} - \overline{r_i}u + \overline{q_i}v}{\overline{p_{i-1}} - \overline{r_{i-1}}u + \overline{q_{i-1}}v} \\ &= (-1)^n \frac{\overline{p_n} - \overline{r_n}u + \overline{q_n}v}{\overline{p_0} - \overline{r_0}u + \overline{q_0}v} \\ &= (-1)^n \frac{\overline{p_n} - \overline{r_n}u + \overline{q_n}v}{v} \end{aligned}$$

This yields the interesting formula

$$(3.7) \quad \overline{p_n} - \overline{r_n}u + \overline{q_n}v = (-1)^n \prod_{i=0}^n v_i.$$

We then have

$$\begin{aligned} d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) &= \frac{|\overline{q_n}v - \overline{r_n}u + \overline{p_n}|^{1/2}}{|q_n|^{1/2}}, \\ &= \frac{|\prod_{i=0}^n v_i|^{1/2}}{|q_n|^{1/2}}. \end{aligned}$$

Noting that $q_n \in \mathbb{Z}[i]$ and that $q_n \neq 0$ by Lemma 3.20 completes the proof. \square

Corollary 3.22. *If $h \in K \setminus \mathbb{H}(\mathbb{Q})$, then $|q_n|$ tends to ∞ .*

Proof. This follows almost immediately from the fact that there are only finitely many rational points $(\frac{r}{q}, \frac{p}{q}) \in \mathcal{S}$ that are written lowest terms, are inside the unit sphere, and have $|q|$ bounded. Since the volume of ϵ -radius balls centered at these points shrinks to zero as ϵ shrinks to zero, no irrational point h can be arbitrarily well approximated by such points. \square

As a corollary to the proof of Theorem 3.21 we obtain a new form of the classical formula for regular continued fractions:

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n(q_{n+1} + q_n \cdot T^{n+1}x)}.$$

The left-hand side of this formula may be considered to be the distance between x and the point p_n/q_n . Recall that in Theorem 3.21 we showed that

$$(3.8) \quad d(\mathbb{K}\{\gamma_i\}_{i=0}^n, h) = \left| v - \left(\frac{r_n}{q_n} \right) u + \left(\frac{p_n}{q_n} \right) \right|^{1/2}.$$

Theorem 3.23. *Let $h \in \mathbb{H}$ with continued fraction digits $CF(h) = \{\gamma_i\}$, associated to a fundamental domain K with $\text{rad}(K) < 1$, and rational approximates $\mathbb{K}\{\gamma_i\}_{i=1}^n = (\frac{r_n}{q_n}, \frac{p_n}{q_n})$. Then, in the notation of Lemma 3.16,*

$$v - \left(\frac{r_n}{q_n} \right) u + \left(\frac{p_n}{q_n} \right) = \frac{1}{\overline{q_n}(q_{n+1} + \tilde{q}_{n+1}u_{n+1} - q_nv_{n+1})}.$$

Proof. This follows immediately from (3.7) and Lemma 3.19. \square

3.5. Uniform convergence. We continue with the assumptions of Theorem 3.21 and the notation of Lemma 3.16. The purpose of this section is to show that the terms $(\frac{\tilde{r}_n}{\tilde{q}_n}, \frac{\tilde{p}_n}{\tilde{q}_n})$ also converge to (u, v) . There are several ways to prove this, including along the lines of the proof of Theorem 3.21, or using the geometric duality between the columns of Q_n . The proof below provides some lemmas of independent interest.

Lemma 3.24. *We have that*

$$\sqrt{2|q_n|} \leq |\tilde{q}_n| \leq \sqrt{2|q_n||q_{n-1}|}$$

Proof. We associate to h a sequence of unitary transformations $A_{\gamma_i} \in U(2, 1; \mathbb{Z}[\mathbf{i}])$. We then have $Q_n = A_{\gamma_1} \cdots A_{\gamma_n} \in U(2, 1; \mathbb{Z}[\mathbf{i}])$. Recall further that for any matrix $M \in U(2, 1)$ we have $M^\dagger \mathbb{J} M = I$. In particular, $U(2, 1)$ is closed under transpose conjugation, so we have $Q_n^\dagger \in U(2, 1; \mathbb{Z}[\mathbf{i}])$. This matrix has (by Lemma 3.16) the form

$$Q_n^\dagger = \overline{\begin{pmatrix} q_n & r_n & p_n \\ \tilde{q}_n & \tilde{p}_n & \tilde{r}_n \\ -q_{n-1} & -r_{n-1} & -p_{n-1} \end{pmatrix}}.$$

Since $Q_n^\dagger \in U(2, 1; \mathbb{Z}[\mathbf{i}])$, we must have that the first column is a null vector with respect to \mathbb{J} . That is, by (2.3) we have

$$|\tilde{q}_n|^2 + 2 \operatorname{Re}(\overline{q_n} q_{n-1}) = 0 \quad \text{and thus} \quad |\tilde{q}_n| = \sqrt{-2|q_n|^2 \operatorname{Re}\left(\frac{q_{n-1}}{q_n}\right)}$$

Since \tilde{q}_n and q_n are non-zero, the real part of q_{n-1}/q_n must be non-zero and so must have norm at least $1/|q_n|$. Also, the absolute value of the real part of q_{n-1}/q_n must be at most $|q_{n-1}/q_n|$. This completes the proof of the inequality. \square

In particular, Lemma 3.24 shows that $|\tilde{q}_n|$ tends to infinity as n grows.

Lemma 3.25. *We have*

$$\frac{\tilde{r}_n}{\tilde{q}_n} = \frac{r_n}{q_n} + \frac{\overline{q_n}}{\tilde{q}_n \cdot q_n} \quad \text{and} \quad \frac{\tilde{p}_n}{\tilde{q}_n} = \frac{p_n}{q_n} - \frac{\overline{r_n}}{\tilde{q}_n \cdot q_n}.$$

Proof. Lemma 3.17 implies that

$$\overline{q_n} = -\tilde{q}_n r_n + \tilde{r}_n q_n.$$

Solving for \tilde{r}_n and dividing by \tilde{q}_n gives the first equality.

In addition, Lemma 3.17 implies that

$$\overline{r_n} = \tilde{p}_n q_n - p_n \tilde{q}_n.$$

Solving for \tilde{p}_n and dividing by \tilde{q}_n gives the second equality. \square

Theorem 3.26. *Under the assumptions of Theorem 3.21 and with the notation of Lemma 3.16, the points $\left(\frac{\tilde{r}_n}{\tilde{q}_n}, \frac{\tilde{p}_n}{\tilde{q}_n}\right)$ converge to $h = (u, v)$, provided h is irrational.*

Proof. Since we know by Theorem 3.21 that $(r_n/q_n, p_n/q_n)$ converges to (u, v) , it suffices, by Lemma 3.25, to show that

$$\frac{\overline{q_n}}{\tilde{q}_n \cdot q_n} \quad \text{and} \quad \frac{\overline{r_n}}{\tilde{q}_n \cdot q_n}$$

converge to 0. This is easy to see for the first number, since $|\overline{q_n}/q_n| = 1$ and $|\tilde{q}_n|$ grows to infinity. For the second number, we have that $|\overline{r_n}/q_n| = |r_n/q_n|$ which converges to $|u|$, and hence is bounded. Since, again $|\tilde{q}_n|$ grows to infinity, the second number must converge to 0 as well. \square

4. HEISENBERG GAUSS MAP

Let K be a fundamental domain for $\mathbb{H}(\mathbb{Z})$ with $\text{rad}(K) < 1$ and $T : K \rightarrow K$ the associated Gauss map (for this section, we will ignore the fact that T is not well-defined at the origin). The purpose of this section is to prove the existence of a measure μ on K , absolutely continuous with respect to Lebesgue measure λ on \mathbb{R}^3 , for which T is invariant, and to show that T is ergodic; that is, $T^{-1}E = E$ implies $\mu(E) = 0$ or $\mu(K \setminus E) = 0$. We will use the following variant of Rényi's theorem, whose terminology we will describe below (see the proof of Lemma 4.9).

Theorem 4.1 (See Theorems 4 and 8 in [15]). *Let T give rise to a fibred system over a set K , with digit set \mathcal{D} . Let λ be some measure. Suppose*

- (1) $\lambda(K) = 1$;
- (2) *All cylinder sets are full*;
- (3) *For any infinite admissible sequence $w = \{w_1, w_2, \dots\}$ of digits from \mathcal{D} , we have*

$$\lim_{n \rightarrow \infty} \text{diam } C_{\{w_1, w_2, \dots, w_n\}} = 0;$$

- (4) *There is a constant $C \geq 1$ such that for all finite admissible strings w of length n ,*

$$\frac{\sup_{y \in T^n C_w} J_y T^n}{\inf_{y \in T^n C_w} J_y T^n} \leq C.$$

Then T is ergodic and admits a finite invariant measure μ , which is absolutely continuous with respect to λ (and vice-versa).

Remark 4.2. In literature on continued fractions, it is customary to consider the inverse transformation to T^n given by $T_w^{-n}(h) = \iota\gamma_1 \cdots \iota\gamma_n h$. In the fourth condition of Theorem 4.1, one then bounds the Jacobian of T^{-n} , with notation

$$\omega(\gamma_1, \dots, \gamma_n; y) := J_y T_w^{-n}.$$

We will not use this notation since we do not manipulate the digits γ_i .

Remark 4.3. The Gauss Map T does not satisfy the conditions of Theorem 4.1. Instead, we will define a related map S that does satisfy the conditions. However, we start by studying T .

Let $w = \{\gamma_1, \dots, \gamma_n\}$ be a sequence of elements of $\mathbb{H}(\mathbb{Z})$, and consider the (possibly empty) *cylinder set* C_w of points in K whose continued fraction expansion starts with w . A cylinder C_w is *full* if $T^n C_w = K$, that is, if by removing the first n digits from points in C_w , we see the full range of points we would expect to see. Note that, for example, the cylinder $C_{\{(1,0)\}}$ is not full for any choice of K . For a point $h \in K$ with $CF(h) = \{\gamma_i\}$, let $n(h)$ denote smallest number n such that the cylinder $C_{\{\gamma_1, \dots, \gamma_n\}}$ is a full cylinder.

Working in the geometric model of \mathbb{H} , let λ denote the Lebesgue measure on $\mathbb{R}^3 = \mathbb{H}$. Note that λ is also the Haar measure for \mathbb{H} , that is, left multiplication by an element of \mathbb{H} does not distort λ .

In the next four lemmas, we study the cylinders C_w . We show that as w grows, the cylinder becomes smaller and, at some point, it becomes full. Furthermore, the

transformation T distorts λ on C_w in a uniform way for all cylinders at the same rank (length of w).

Lemma 4.4. *Let $h \in K$ be an irrational point contained on the interior of K and $CF(h) = \{\gamma_i\}_{i=1}^\infty$. Denote the cylinder sets associated to h by $C_n = C_{\{\gamma_1, \dots, \gamma_n\}}$. For sufficiently high n , we have $C_n \subset K$. Furthermore,*

$$\lim_{n \rightarrow \infty} \text{diam}(C_n) = 0 \qquad \lim_{n \rightarrow \infty} \lambda(C_n) = 0.$$

Proof. By Theorem 3.21, we have that the rational approximates of h converge to h . In particular, the rational approximates $\mathbb{K}\{\gamma_i\}_{i=1}^n$ are eventually contained in K , and therefore in the associated cylinder set C_n . Furthermore, our bounds on the Euclidean distance (in the planar Siegel model) from $\mathbb{K}\{\gamma_i\}_{i=1}^n$ to any point of C_n are independent of the particular point. Thus, we have that for sufficiently large n , the closures $\overline{C_n}$ of the cylinder sets form a nested sequence of compact sets whose intersection is equal to $\{h\}$. This implies that the sets C_n , in the limit, become trivial both in diameter and measure, as desired. \square

Lemma 4.5. *Under the assumptions of Lemma 4.4, there exists $n(h) \in \mathbb{N}$ such that the cylinder set $C_{n(h)}$ is full.*

Proof. Recall that a cylinder set C_n is full if $T^n C_n = K$. Recall that for a point in C_n , the map T^n is given by $T^n(h') = \gamma_n^{-1} \iota \cdots \gamma_1^{-1} \iota h'$. Thus, C_n is full exactly if $C_n = \iota \gamma_1 \cdots \iota \gamma_n K$. By definition of C_n , we have $C_n = K \cap \iota \gamma_1 \cdots \iota \gamma_n K$, so it suffices to show that for high n the set $\iota \gamma_1 \cdots \iota \gamma_n K$ does not contain points outside of K .

Now, by Lemma 4.4, for high n , C_n is contained on the interior of K . Furthermore, it is easy to see that the set $\iota \gamma_1 \cdots \iota \gamma_n K$ is connected. Thus, for sufficiently high n , $\iota \gamma_1 \cdots \iota \gamma_n K$ cannot have points outside of K , and C_n must be a full cylinder. \square

Recall that the Jacobian $J_h f$ of a measureable transformation f measures the volume distortion caused by f at the point h .

Lemma 4.6. *Let h be an irrational interior point of K with $CF(h) = \{\gamma_i\}_{i=1}^\infty$. Let $w = \{\gamma_1, \dots, \gamma_n\}$ be a finite subsequence of the continued fraction digits. We then have*

$$J_h T^n = \prod_{i=0}^{n-1} \|T^i(h)\|^{-8}.$$

Proof. The transformation is given by $T^n = \gamma_n^{-1} \iota \cdots \gamma_1^{-1} \iota$. Left translation by γ_i acts by a shear on \mathbb{H} in the geometric model, and therefore does not distort Lebesgue measure. It suffices to show that for any point $h \in K$,

$$J_h \iota = \|h\|^{-8}.$$

Recall that by Lemma 2.1 we have

$$d(\iota h, \iota k) = \frac{d(h, k)}{\|h\| \|k\|}.$$

Thus, the image of a ball of radius ϵ around h is, in the limit, a ball of radius $\|h\|^{-2} \epsilon$. By Lemma 2.3, this implies a distortion factor of $\|h\|^{-8}$. \square

Lemma 4.7. *Let $w = \{\gamma_1, \dots, \gamma_n\}$ be a sequence of elements of $\mathbb{H}(\mathbb{Z})$. Then,*

$$\frac{\sup_{h \in C_w} J_h T^n}{\inf_{h \in C_w} J_h T^n} \leq \exp \left(\frac{16 \operatorname{rad}(K)}{1 - \operatorname{rad}(K)^2} \right)$$

Proof. Let $h, k \in C_w$. By Lemma 4.6, we have

$$\frac{J_h T^n}{J_k T^n} = \left(\prod_{i=0}^{n-1} \frac{\|T^i k\|}{\|T^i h\|} \right)^8.$$

By the triangle inequality we have:

$$\frac{\|T^i k\|}{\|T^i h\|} \leq 1 + \frac{d(T^i h, T^i k)}{\|T^i h\|}.$$

Furthermore, for $i \neq n$,

$$\begin{aligned} \frac{d(T^i h, T^i k)}{\|T^i h\|} &= \frac{d(\iota T^i h, \iota T^i k)}{\|\iota T^i k\|} = \frac{d(T^{i+1} h, T^{i+1} k)}{\|\iota T^i k\|} \\ &\leq d(T^{i+1} h, T^{i+1} k) \operatorname{rad}(K) \leq \frac{d(T^{i+1} h, T^{i+1} k)}{\|T^{i+1} h\|} \operatorname{rad}(K)^2. \end{aligned}$$

Continuing inductively, we have that

$$\frac{d(T^i h, T^i k)}{\|T^i h\|} \leq d(T^n h, T^n k) \operatorname{rad}(K)^{2(n-i)} \leq 2 \operatorname{rad}(K)^{2(n-i)+1}.$$

Returning to the Jacobian, we have

$$\begin{aligned} \frac{J_h T^n}{J_k T^n} &= \left(\prod_{i=0}^{n-1} \frac{\|T^i k\|}{\|T^i h\|} \right)^8 \leq \prod_{i=0}^{n-1} \left(1 + 2 \operatorname{rad}(K)^{2(n-i)+1} \right)^8 \\ &\leq \exp \left(8 \sum_{i=0}^{n-1} 2 \operatorname{rad}(K)^{2(n-i)+1} \right) \leq \exp \left(8 \sum_{i=1}^n 2 \operatorname{rad}(K)^{2i+1} \right) \\ &\leq \exp \left(16 \sum_{i=1}^{\infty} \operatorname{rad}(K)^{2i+1} \right) = \exp \left(\frac{16 \operatorname{rad}(K)}{1 - \operatorname{rad}(K)^2} \right), \end{aligned}$$

as desired. \square

Definition 4.8. Let the *jump transformation* $S : K \rightarrow K$ associated to T be defined by $S(h) = T^{n(h)}(h)$. Recall that $n(h)$ is the smallest integer so that C_n is a full cylinder.

By Lemmas 4.4 and 4.5, S is well-defined for λ -almost-every point of K .

Lemma 4.9. *The jump transformation S is ergodic and admits an invariant measure λ_S , which is absolutely continuous with respect to λ .*

Proof. It suffices to show that the jump transformation and its associated collection of cylinders satisfy the conditions of Theorem 4.1.

The map S is defined on the fundamental domain K of $\mathbb{H}(\mathbb{Z})$. By Lemma 2.17, we have that $\lambda(K) = 1$.

The *fibred system* associated to S arises from the collection of full cylinders $C_{n(h)}$ for points $h \in K$ (these are cylinders with respect to T). (Note that if $k \in C_{n(h)}$,

then $n(k) = n(h)$ and so $C_{n(k)} = C_{n(h)}$.) On these cylinders, the maps $T^{n(h)}$ and S agree, so the cylinders are also full with respect to S .

By Lemma 4.4, the diameters of the cylinder sets C_w limit to 0 as the length of w increases.

For the last condition of Theorem 4.1, we may apply Lemma 4.7. While the lemma is stated for T , it also holds for S since $S(h) = T^{n(h)}(h)$. \square

Theorem 4.10. *Let K be a fundamental domain for $\mathbb{H}(\mathbb{Z})$ with $\text{rad}(K) < 1$, and $T : K \rightarrow K$ the associated Gauss map. Then T is ergodic and admits an invariant measure λ_T , which is absolutely continuous (in fact, equivalent to) Lebesgue measure λ .*

Proof. In Lemma 4.9 we used a variant of Rényi's Theorem (Theorem 4.1) to show that the jump transformation associated to T is ergodic and admits an invariant measure λ_S which is equivalent to λ .

Let $D_0 = K$, and for each $n > 0$ let D_n denote the union of non-full cylinders of rank n :

$$D_n = \bigcup C_w, \quad w = \{\gamma_1, \dots, \gamma_n\} \text{ and } C_w \text{ is not full.}$$

Theorem 11 in [15] states that T is ergodic with respect to an invariant measure λ_T given by

$$\lambda_T(E) = \sum_{n=0}^{\infty} \lambda_S(T^{-n}E \cap D_n).$$

Since we have that λ_S is equivalent to Lebesgue measure λ , it follows directly that the same is true for λ_T . \square

In the case of classical continued fractions, the density of the invariant measure is given by a relatively simple expression. It's not clear whether this is the case for Heisenberg continued fractions, but it seems likely that the density is, at the very least, given by a real-analytic function. One may attempt a proof along the lines of Theorem 6.1.3 of [12].

As an application of the ergodicity of T , we can attempt to visualize the invariant measures on K_C and K_D . Specifically, the Birkhoff Ergodic Theorem states that for a generic point $h \in K$, the time average of $T^n(h)$ estimates the associated invariant measure. The results are displayed in Figure 4.

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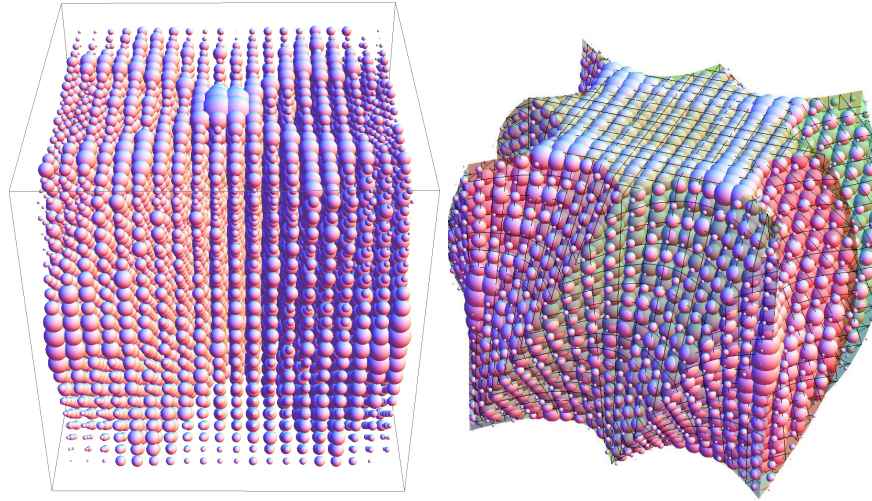


FIGURE 4. The Gauss-Kuzmin measure on the sets K_C (left) and K_D (right). The location of the n^{th} iterate of $h_0 = (\pi - 3, e - 3, 0)$ under the corresponding Gauss map was tracked for 5,000,000 iterations. In the illustration, the size of the sphere demonstrates the time spent by the iterates in its vicinity.

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